Application of Integrals

Let f(x) be a function defined on the interval [a, b] and F(x) be its anti-derivative. Then,

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The above is called the second fundamental theorem of calculus.

$\int_a^b f(x)\,dx$ is defined as the definite integral of f(x) from x = a to x = b. The numbers and b are called limits of integration. We write

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

Evaluation of Definite Integrals by Substitution

Consider a definite integral of the following form

$$\int_a^b f[g(x)] \cdot g'(x)\,dx$$

Step 1 Substitute g(x) = t

⇒g ‘(x) dx = dt

Step 2 Find the limits of integration in new system of variable i.e..the lower limit is g(a) and the upper limit is g(b) and the g(b) integral is now $\int_{g(a)}^{g(b)} f(t)\,dt.$

Step 3 Evaluate the integral, so obtained by usual method.

Properties of Definite Integral

1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$

Generalization

If $a < c_1 < c_2 < \ldots < c_{n-1} < c_n < b$, then

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx + \ldots + \int_{c_{n-1}}^{c_n} f(x)\,dx + \int_{c_n}^{b} f(x)\,dx$$

4. $\int_a^a f(x)\,dx = 0$

5. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Deduction $\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \frac{a}{2}$

6. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Deduction $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$

7. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

8. $\int_{-a}^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(-x)\,dx$

9. $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx \text{ if, } f(2a-x) = f(x) \\ 0, \quad \text{if } f(2a-x) = -f(x) \end{cases}$

or

$$\int_a^b f(x)\,dx = \begin{cases} 0, \text{ if } f(a+x) = -f(b-x) \\ 2\int_a^{\frac{a+b}{2}} f(x)\,dx, \text{ if } f(a+x) = f(b-x) \end{cases}$$

10. $\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, \text{ if } f(x) \text{ is even } i.e., f(-x) = f(x) \\ 0, \quad \text{if } f(x) \text{ is odd } i.e., f(-x) = -f(x) \end{cases}$

11. If $\int_a^b f(x)\,dx = (b-a)\int_0^1 f[(b-a)x + a]\,dx$

12. If $f(x)$ is periodic function with period T, [*i.e.*, $f(x+T) = f(x)$]
Then, $\int_a^{a+T} f(x)\,dx$ is independent of a.

Summation of Series by Definite Integral

We know that,

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} h \sum_{r=1}^{n} f(a + rh)$$

where, $nh = b - a$

Now, put $a = 0, b = 1$

$\therefore$ $nh = 1 - 0 = 1$ or $h = \frac{1}{n}$

$$\therefore \quad \int_0^1 f(x)\,dx = \lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)$$

Method Express the given series in the form of

$$\lim_{n \to \infty} \sum \frac{1}{n} f\left(\frac{r}{n}\right)$$

Replace $\frac{r}{n}$ by x and $\frac{1}{n}$ by dx and the limit of the sum is $\int_0^1 f(x)\,dx$.

Note $\lim_{n\to\infty} \sum_{r=1}^{pn} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_\alpha^\beta f(x)\,dx$

where, $\alpha = \lim_{n\to\infty} \frac{r}{n} = 0$ (as $r = 1$)

and $\beta = \lim_{n\to\infty} \frac{r}{n} = p$ (as $r = pn$)

The method to evaluate the integral, as limit of the sum of an infinite series is known as integration by first principle.

Important Results

(i) (a) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x} dx$

(b) $\int_0^{\pi/2} \frac{\tan^n x}{1 + \tan^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{dx}{1 + \tan^n x}$

(c) $\int_0^{\pi/2} \frac{dx}{1 + \cot^n x} = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cot^n x}{1 + \cot^n x} dx$

(d) $\int_0^{\pi/2} \frac{\tan^n x}{\tan^n x + \cot^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cot^n x}{\tan^n x + \cot^n x} dx$

(e) $\int_0^{\pi/2} \frac{\sec^n x}{\sec^n x + \mathrm{cosec}^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\mathrm{cosec}^n x}{\sec^n x + \mathrm{cosec}^n x} dx$ where, $n \in R$

(ii) $\int_0^{\pi/2} \frac{a^{\sin^n x}}{a^{\sin^n x} + a^{\cos^n x}} dx = \int_0^{\pi/2} \frac{a^{\cos^n x}}{a^{\sin^n x} + a^{\cos^n x}} dx = \frac{\pi}{4}$

(iii) (a) $\int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x \, dx = -\frac{\pi}{2} \log 2$

(b) $\int_0^{\pi/2} \log \tan x \, dx = \int_0^{\pi/2} \log \cot x \, dx = 0$

(c) $\int_0^{\pi/2} \log \sec x \, dx = \int_0^{\pi/2} \log \mathrm{cosec} x \, dx = \frac{\pi}{2} \log 2$

(iv) (a) $\int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2}$

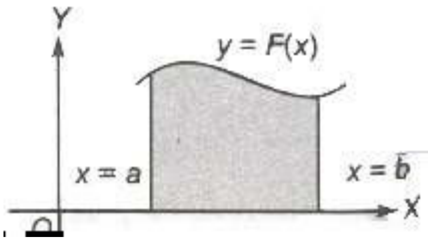
(b) $\int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2}$

(c) $\int_0^{\infty} e^{-ax} x^n \, dx = \frac{n!}{a^n + 1}$

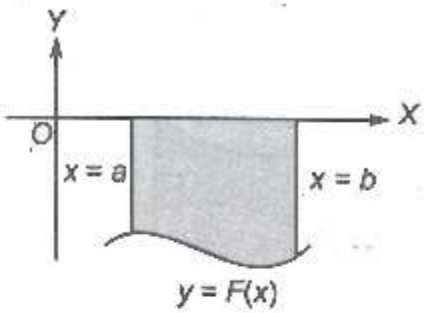
Area of Bounded Region

The space occupied by the curve along with the axis, under the given condition is called area of bounded region.

(i) The area bounded by the curve y = F(x) above the X-axis and between the lines x = a, x = b is given by $\int_a^b y \, dx = \int_a^b F(x) \, dx$



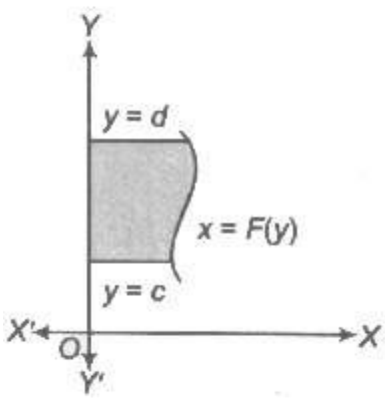
(ii) If the curve between the lines x = a, x = b lies below the X-axis, then the required area is given by



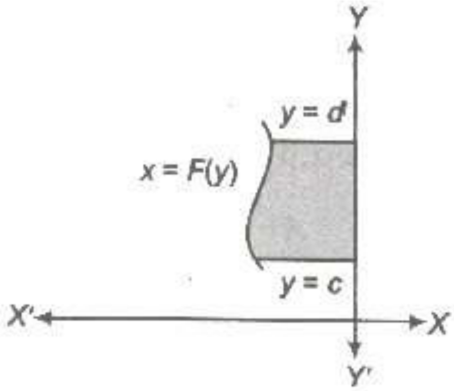
$$\left|\int_a^b (-y)\,dx\right| = \left|-\int_a^b y\,dx\right| = \left|-\int_a^b F(x)\,dx\right|$$

(iii) The area bounded by the curve x = F(y) right to the Y-axis and the lines y = c, y = d is given by

$$\int_c^d x\,dy = \int_c^d F(y)\,dy$$



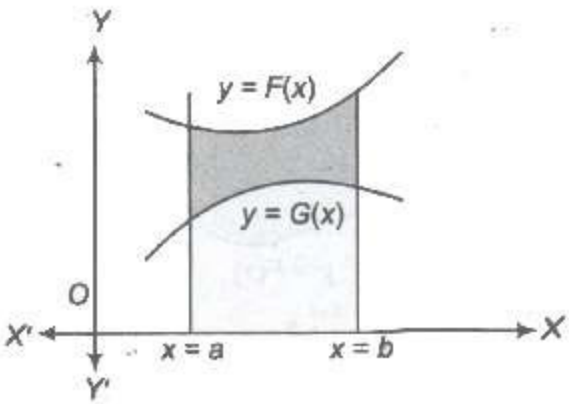
(iv) If the curve between the lines y = c, y = d left to the Y-axis, then the area is given by



$$\left|\int_c^d (-x)\, dy\right| = \left|-\int_c^d x\, dy\right|$$

$$= \left|-\int_c^d F(y)\, dy\right|$$

(v) Area bounded by two curves y = F (x) and y = G (x) between x = a and x = b is given by



$$\int_a^b \{F(x) - G(x)\}\, dx$$

(vi) Area bounded by two curves x = f(y) and x = g(y) between y=c and y=d is given by $\int_c^d [F(y) - G(y)]\, dy$

(vii) If F (x) ≥. G (x) in [a, c] and F (x) ≤ G (x) in [c,d], where a < c < b, then area of the region bounded by the curves is given as

$$\text{Area} = \int_a^c \{F(x) - G(x)\}\, dx + \int_c^b \{G(x) - F(x)\} dx$$

MCQS

1. The area bounded by the circle $x^2+y^2=2$ is equal to :

(a) sq. units (b) sq. units (c) 4 sq. units (d) sq. units

2. The area of the region bounded by the curve $y=x^2$ and the line $y=16$ is

(a) (b) (c) (d)

3. The area of the region bounded by the parabola $y^2=x$ and the straight line $2y=x$ is

(a) sq. units (b) 1 sq. unit (c) sq. units (d) sq. unit

4. The area of the region bounded by the curve $y=\sin x$ between the ordinate and the x-axis is:

(a) 2 sq. units (b) 4 sq. units (c) 3 sq. units (d) 1 sq. unit

5. The area of the region bounded by the curve and x-axis is

(a) 8 sq. units (b) 20 sq. units (c) 16 sq. units (d) 256 sq. units

6. The area bounded by the curve $y=x^2$ the x- axis and the line $x=2^{1/3}$ is divided into two equal areas by the line $x=k$. The value of k is

(a) $2^{1/3}-1$ (b) 1 (c) $2^{-2/3}$ (d) $2^{-1/3}$

7. The area $\{ (x,y):|x|\geq y\geq x^2\}$ is equal to:

(a) (b) (c) (d) None of these

8. The area is equal to:

(a) (b) (c) (d) None of these

9. The area enclosed between the curves $y=x^2$ and $x=y^2$ is:

(a) (b) (c) (d) None of these

10. The area enclosed between the curve $y^2=4x$ and the line $y=x$ is

(a) (b) (c) (d)

11. The area enclosed between the curve $y=x^{1/3}$, the y-axis and the lines $y=-1$, $y=1$ is

(a) 0 (b) (c) (d) None of these

12. The area of a parabola $y^2=4ax$ bounded by its latus rectum is:

(a) (b) (c) (d)

13. The area between the hyperbola $xy=c^2$, the x-axis and the ordinates at a and b with a>b is:

(a) (b) (c) (d) None of these

14. The area of the region bounded by the curve $y=2x-x^2$ and the line y=x is

(a) (b) (c) (d)

15. The area enclosed by the ellipse is

(a) (b) (c) a^2b (d) ab^2

16. if the area bounded by the curve y=f(x) the coordinate axes and the line is given by , then f(x) is equal to

(a) xe^x-e^x (b) xe^x+e^x (c) e^x (d) xe^x

17. The area between and st. line

(a) (b) (c) (d)

18. The area of the figure bounded by y=sin x, y=cos x in the first quadrant is

(a) (b) (c) (d) None of these

19. The area bounded by the curves $y=xe^x$,$y=xe^{-x}$ and the line x=1 is:

(a) (b) (c) (d)

20. The area bounded by the curve $y=x^3$, x-axis and two ordinates x=1 and x=2 is equal to:

(a) sq. units (b) sq. units (c) sq. units (d) sq. units

21. The area bounded by the curve $y=4x-x^2$ and the x-axis is :

(a) sq. units (b) sq. units (c) sq. units (d) sq. units

22. Area between the curve $y=4+3x-x^2$ and x-axis in square units is:

(a) (b) (c) (d) None of these

23. Area bounded by the curve y= x and x-axis between x=0 and x=2 is:

(a) (b) (c) (d) None of these

24. The area bounded by the curve $y=2x-x^2$ and the straight line y=-x is given by

(a) (b) (c) (d) None of these

25. The area enclosed between the curve y= $\log_e(x+e)$ and the co-ordinate axes is:

(a) 2 (b) 1 (c) 4 (d) 3

ANSWERS

1	**d**	2	**b**	3	**a**	4	**d**	5	**a**	6	**b**	7	**b**	8	**a**	9	**a**
10	**a**	11	**b**	12	**a**	13	**b**	14	**a**	15	**c**	16	**b**	17	**a**	18	**d**
19	**c**	20	**b**	21	**c**	22	**c**	23	**c**	24	**a**	25	**b**				