CONTINUITY

1. DEFINITION

A function $f(x)$ is said to be continuous at x = a; where a $\in$ domain of $f(x)$, if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

i.e., LHL = RHL = value of a function at x = a

or $\lim_{x\to a} f(x) = f(a)$

1.1 Reasons of discontinuity

If $f(x)$ is not continuous at x = a, we say that $f(x)$ is discontinuous at x = a.

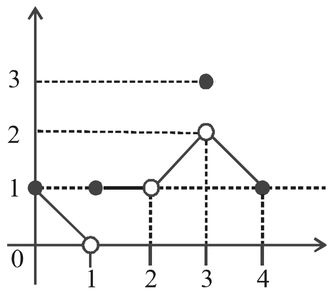
There are following possibilities of discontinuity :

1. $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist but they are not equal.

2. $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exists and are equal but not equal to $f(a)$.

3. $f(a)$ is not defined.

4. At least one of the limits does not exist. Geometrically, the graph of the function will exhibit a break at the point of discontinuity.



The graph as shown is discontinuous at x = 1, 2 and 3.

2. PROPERTIES OF CONTINUOUS FUNCTIONS

Let $f(x)$ and $g(x)$ be continuous functions at x = a. Then,

1. $c\,f(x)$ is continuous at x = a, where c is any constant.

2. $f(x) \pm g(x)$ is continuous at x = a.

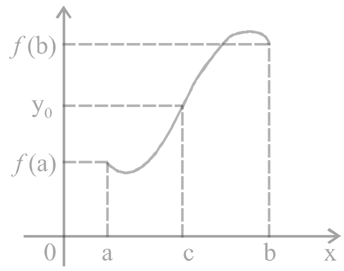
3. $f(x).\,g(x)$ is continuous at x = a.

4. $f(x)/g(x)$ is continuous at x = a, provided $g(a) \neq 0$.-

5. If $f(x)$ is continuous on [a, b] such that $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one solution of equation $f(x) = 0$ in the open interval (a, b).

3. THE INTERMEDIATE VALUE THEOREM

Suppose $f(x)$ is continuous on an interval I, and a and b are any two points of I. Then if y_0 is a number between $f(a)$ and $f(b)$, their exits a number c between a and b such that $f(c) = y_0$.



The Function f, being continuous on (a,b) takes on every value between $f(a)$ and $f(b)$

Note...

That a function f which is continuous in [a, b] possesses the following properties :

(i) If $f(a)$ and $f(b)$ possess opposite signs, then there exists at least one solution of the equation $f(x) = 0$ in the open interval (a, b).

(ii) If K is any real number between $f(a)$ and $f(b)$, then there exists at least one solution of the equation $f(x) = K$ in the open interval (a, b).

4. CONTINUITY IN AN INTERVAL

(a) A function f is said to be continuous in (a, b) if f is continuous at each and every point $\in$ (a, b).

(b) A function f is said to be continuous in a closed interval [a, b] if :

(1) f is continuous in the open interval (a, b) and

(2) f is right continuous at 'a' i.e. $\text{Limit}_{x\to a^+} f(x) = f(a)$ = a finite quantity.

(3) f is left continuous at 'b'; i.e. $\text{Limit}_{x\to b^-} f(x) = f(b)$ = a finite quantity.

5. A LIST OF CONTINUOUS FUNCTIONS

	Function f (x)	Interval in which f (x) is continuous
1.	constant c	$(-\infty, \infty)$
2.	x^n, n is an integer ≥ 0	$(-\infty, \infty)$
3.	x^{-n}, n is a positive integer	$(-\infty, \infty) - \{0\}$
4.	$\lvert x-a \rvert$	$(-\infty, \infty)$
5.	$P(x) = a_0x^n + a_1x^{n-1} + ... + a_n$	$(-\infty, \infty)$
6.	$\frac{p(x)}{q(x)}$, where p (x) and q (x) are polynomial in x	$(-\infty, \infty) - \{x ; q(x) = 0\}$
7.	sin x	$(-\infty, \infty)$
8.	cos x	$(-\infty, \infty)$
9.	tan x	$(-\infty, \infty) - \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$
10.	cot x	$(-\infty, \infty) - \{n\pi : n \in I\}$
11.	sec x	$(-\infty, \infty) - \{(2n+1)\pi/2 : n \in I\}$
12.	cosec x	$(-\infty, \infty) - \{n\pi : n \in I\}$
13.	e^x	$(-\infty, \infty)$
14.	$\log_e x$	$(0, \infty)$

6. TYPES OF DISCONTINUITIES

Type-1 : (Removable type of discontinuities)

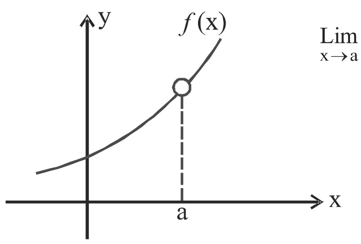
In case, $\underset{x \to c}{\text{Limit}}\ f(x)$ exists but is not equal to $f(c)$ then the function is said to have a **removable discontnuity or discontinuity of the first kind.** In this case, we can redefine the function such that $\underset{x \to c}{\text{Limit}}\ f(x) = f(c)$ and make it continuous at x = c. Removable type of discontinuity can be further classified as :

(a) Missing Point Discontinuity :

Where $\underset{x \to a}{\text{Limit}}\ f(x)$ exists finitely but $f(a)$ is not defined.

E.g. $f(x) = \frac{(1-x)(9-x^2)}{(1-x)}$ has a missing point discontinuity at x = 1, and

$f(x) = \frac{\sin x}{x}$ has a missing point discontinuity at x = 0.



$\underset{x \to a}{\text{Lim}}\ f(x) \to$ exist finitely.
$f(a) \to$ does not exist.

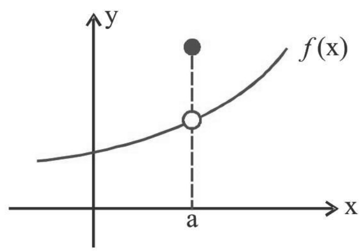
missing point discontinuity at x = a

(b) Isolated Point Discontinuity :

Where $\underset{x \to a}{\text{Limit}}\ f(x)$ exists & $f(a)$ also exists but; $\underset{x \to a}{\text{Limit}} \neq f(a)$.

E.g. $f(x) = \frac{x^2 - 16}{x - 4}$, $x \neq 4$ and $f(4) = 9$ has an isolated point discontinuity at x = 4.

Similarly $f(x) = [x] + [-x] = \begin{bmatrix} 0 & \text{if } x \in I \\ -1 & \text{if } x \notin I \end{bmatrix}$ has an isolated point discontinuity at all $x \in I$.

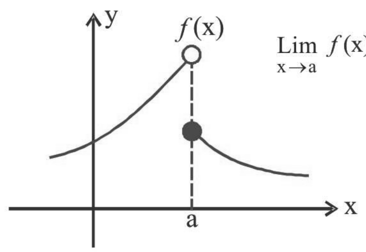


$\underset{x \to a}{\text{Lim}}\ f(x) \to$ exists finitely
$f(a) \to$ exists.
But, $\underset{x \to a}{\text{Lim}}\ f(x) \neq f(a)$

Isolated point discontinuity at x = a

Type-2 : (Non-Removable type of discontinuities)

In case, $\underset{x \to a}{\text{Limit}}\ f(x)$ does not exist, then it is not possible to make the function continuous by redefining it. Such discontinuities are known as **non-removable discontinuity or discontinuity of the 2nd kind.** Non-removable type of discontinuity can be further classified as :



$\underset{x \to a}{\text{Lim}}\ f(x) \to$ does not exist.

non-removable discontinuity at x = a

(a) Finite Discontinuity :

E.g., $f(x) = x - [x]$ at all integral x; $f(x) = \tan^{-1}\frac{1}{x}$ at $x = 0$ and

$f(x) = \dfrac{1}{1 + 2^{\frac{1}{x}}}$ at $x = 0$ (note that $f(0^+) = 0$; $f(0^-) = 1$)

(b) Infinite Discontiunity :

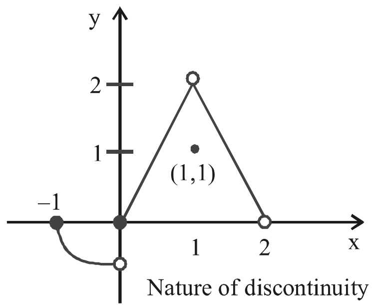
E.g., $f(x) = \dfrac{1}{x-4}$ or $g(x) = \dfrac{1}{(x-4)^2}$ at $x = 4$; $f(x) = 2^{\tan x}$

at $x = \dfrac{\pi}{2}$ and $f(x) = \dfrac{\cos x}{x}$ at $x = 0$.

(c) Oscillatory Discontinuity :

E.g., $f(x) = \sin\dfrac{1}{x}$ at $x = 0$.

In all these cases the value of $f(a)$ of the function at x = a (point of discontinuity) may or may not exist but $\text{Limit}_{x \to a}$ does not exist.



Nature of discontinuity

From the adjacent graph note that

– f is continuous at $x = -1$

– f has isolated discontinuity at x = 1

– f has missing point discontinuity at x = 2

– f has non-removable (finite type) discontinity at the origin.

Note...

(a) In case of dis-continuity of the second kind the non-negative difference between the value of the RHL at x = a and LHL at x = a is called the **jump of discontinuity.** A function having a finite number of jumps in a given interval I is called a piece wise continuous or sectionally continuous function in this interval.

(b) All Polynomials, Trigonometrical functions, exponential and Logarithmic functions are continuous in their domains.

(c) If $f(x)$ is continuous and $g(x)$ is discontinuous at x = a then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at x = a. e.g.

$$f(x) = x \text{ and } g(x) = \begin{cases} \sin\dfrac{\pi}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(d) If $f(x)$ and $g(x)$ both are discontinuous at x = a then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at x = a . e.g.

$$f(x) = -g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

(e) Point functions are to be treated as discontinuous eg. $f(x) = \sqrt{1-x} + \sqrt{x-1}$ is not continuous at x = 1.

(f) A continuous function whose domain is closed must have a range also in closed interval.

(g) If f is continuous at $x = a$ and g is continuous at $x = f(a)$ then the composite $g[f(x)]$ is continous at $x = a$ E.g $f(x) = \dfrac{x\sin x}{x^2+2}$ and $g(x) = |x|$ are continuous at x = 0, hence the composite $(gof)(x) = \left|\dfrac{x\sin x}{x^2+2}\right|$ will also be continuous at x = 0.

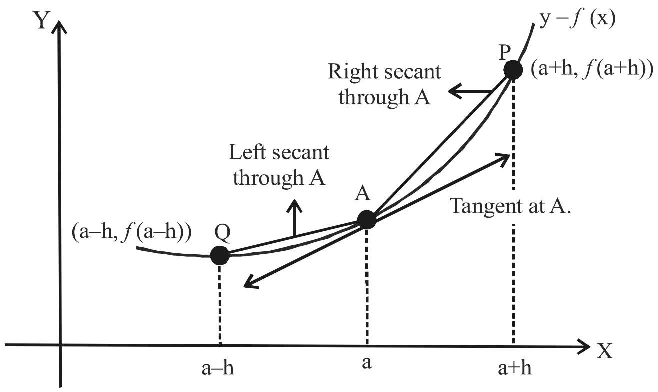
DIFFERENTIABILITY

1. DEFINITION

Let $f(x)$ be a real valued function defined on an open interval (a, b) where $c \in (a, b)$. Then $f(x)$ is said to be differentiable or derivable at x = c,

iff, $\lim\limits_{x \to c} \frac{f(x)-f(c)}{(x-c)}$ exists finitely.

This limit is called the derivative or differentiable coefficient of the function $f(x)$ at x = c, and is denoted by $f'(c)$ or $\frac{d}{dx}(f(x))_{x=c}$.



- Slope of Right hand secant = $\frac{f(a+h)-f(a)}{h}$ as $h \to 0$, $P \to A$ and secant (AP) $\to$ tangent at A

$\Rightarrow$ Right hand derivative $= \lim\limits_{h \to 0}\left(\frac{f(a+h)-f(a)}{h}\right)$

= Slope of tangent at A (when approached from right) $f'(a^+)$.

- Slope of Left hand secant = $\frac{f(a-h)-f(a)}{-h}$ as $h \to 0$, $Q \to A$ and secant AQ $\to$ tangent at A

$\Rightarrow$ Left hand derivative $= \lim\limits_{h \to 0}\left(\frac{f(a-h)-f(a)}{-h}\right)$

= Slope of tangent at A (when approached from left) $f'(a^-)$.

Thus, $f(x)$ is differentiable at x = c.

$$\Leftrightarrow \lim_{\to c} \frac{f(\ )-f(c)}{(\ -c)} \text{ exists finitely}$$

$$\Leftrightarrow \lim_{\to c^-} \frac{f(\ )-f(c)}{(\ -c)} = \lim_{\to c^+} \frac{f(\ )-f(c)}{(\ -c)}$$

$$\Leftrightarrow \lim_{h \to 0} \frac{f(c-h)-f(c)}{-h} = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$$

Hence, $\lim\limits_{x \to c^-} \frac{f(x)-f(c)}{(x-c)} = \lim\limits_{h \to 0} \frac{f(c-h)-f(c)}{-h}$ is called the **left hand derivative** of $f(x)$ at x = c and is denoted by $f'(c^-)$ or $Lf'(c)$.

While, $\lim\limits_{x \to c^+} \frac{f(x)-f(c)}{x-c} = \lim\limits_{h \to 0} \frac{f(c+h)-f(c)}{h}$ is called the **right hand derivative** of $f(x)$ at x = c and is denoted by $f'(c^+)$ or $Rf'(c)$.

If $f'(c^-) \neq f'(c^+)$, we say that $f(x)$ is not differentiable at x = c.

2. DIFFERENTIABILITY IN A SET

1. A function $f(x)$ defined on an open interval (a, b) is said to be differentiable or derivable in open interval (a, b), if it is differentiable at each point of (a, b).

2. A function $f(x)$ defined on closed interval [a, b] is said to be differentiable or derivable. "If f is derivable in the open interval (a, b) and also the end points a and b, then f is said to be derivable in the closed interval [a, b]".

i.e., $\lim\limits_{\to a^+} \frac{f(\ )-f(a)}{-a}$ and $\lim\limits_{\to b^-} \frac{f(\ )-f(b)}{-b}$, both exist.

A function f is said to be a differentiable function if it is differentiable at every point of its domain.

Note...

1. If $f(x)$ and $g(x)$ are derivable at x = a then the functions $f(x)+g(x)$, $f(x)-g(x)$, $f(x).\,g(x)$ will also be derivable at x = a and if $g(a) \neq 0$ then the function $f(x)/g(x)$ will also be derivable at x = a.
2. If $f(x)$ is differentiable at x = a and $g(x)$ is not differentiable at x = a, then the product function $F(x) = f(x).\,g(x)$ can still be differentiable at x = a. E.g. $f(x) = x$ and $g(x) = |x|$.
3. If $f(x)$ and $g(x)$ both are not differentiable at x = a then the product function; $F(x) = f(x).\,g(x)$ can still be differentiable at x = a. E.g., $f(x) = |x|$ and $g(x) = |x|$.
4. If $f(x)$ and $g(x)$ both are not differentiable at x = a then the sum function $F(x) = f(x) + g(x)$ may be a differentiable function. E.g., $f(x) = |x|$ and $g(x) = -|x|$.
5. If $f(x)$ is derivable at x = a

   ⇒ $f'(x)$ is continuous at x = a.

   $$\text{e.g.}\quad f(x) = \begin{cases} {}^{2}\sin\frac{1}{} & \text{if } \neq 0 \\ 0 & \text{if } = 0 \end{cases}$$

3. RELATION B/W CONTINUITY & DIFFERENNINBILINY

In the previous section we have discussed that if a function is differentiable at a point, then it should be continuous at that point and a discontinuous function cannot be differentiable. This fact is proved in the following theorem.

Theorem : If a function is differentiable at a point, it is necessarily continuous at that point. But the converse is not necessarily true,

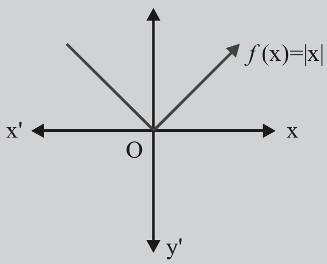
or $f(x)$ is differentiable at x = c

⇒ $f(x)$ is continuous at x = c.

Note...

Converse : The converse of the above theorem is not necessarily true i.e., a function may be continuous at a point but may not be differentiable at that point.

E.g., The function $f(x) = |x|$ is continuous at x = 0 but it is not differentiable at x = 0, as shown in the figure.



The figure shows that sharp edge at x = 0 hence, function is not differentiable but continuous at x = 0.

Note...

(a) Let $f'^{+}(a) = p$ & $f'^{-}(a) = q$ where p & q are finite then :

(i) p = q $\Rightarrow f$ is derivable at x = a

$\Rightarrow f$ is continuous at x = a.

(ii) $p \neq q$ $\Rightarrow f$ is not derivable at x = a.

It is very important to note that f may be still continuous at x = a.

In short, for a function f :

Differentiable $\Rightarrow$ Continuous;

Not Differentiable $\nRightarrow$ Not Continuous

(i.e., function may be continuous)

But,

Not Continuous $\Rightarrow$ Not Differentiable.

(b) If a function f is not differentiable but is continuous at x = a it geometrically implies a sharp corner at x = a.

Theorem 2 : Let f and g be real functions such that fog is defined if g is continuous at x = a and f is continuous at g (a), show that fog is continuous at x = a.

DIFFERENTIATION

1. DEFINITION

(a) Let us consider a function $y = f(x)$ defined in a certain interval. It has a definite value for each value of the independent variable x in this interval.

Now, the ratio of the increment of the function to the increment in the independent variable,

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Now, as $\Delta x \to 0$, $\Delta y \to 0$ and $\frac{\Delta y}{\Delta x} \to$ finite quantity, then derivative $f(x)$ exists and is denoted by y' or $f'(x)$ or $\frac{dy}{dx}$

Thus, $f'(x) = \lim_{x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

(if it exits)

for the limit to exist,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h}$$

(Right Hand derivative) (Left Hand derivative)

(b) The derivative of a given function f at a point x = a of its domain is defined as :

$\operatorname{Limit}_{h \to 0} \frac{f(a+h) - f(a)}{h}$, provided the limit exists & is denoted by $f'(a)$.

Note that alternatively, we can define

$f'(a) = \operatorname{Limit}_{x \to a} \frac{f(x) - f(a)}{x - a}$, provided the limit exists.

This method is called first principle of finding the derivative of f(x).

2. DERIVATIVE OF STANDARD FUNCTION

(i) $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$; $x \in R, n \in R, x > 0$

(ii) $\frac{d}{dx}(e^x) = e^x$

(iii) $\frac{d}{dx}(a^x) = a^x \cdot \ln a \; (a > 0)$

(iv) $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

(v) $\frac{d}{dx}(\log_a |x|) = \frac{1}{x}\log_a e$

(vi) $\frac{d}{dx}(\sin x) = \cos x$

(vii) $\frac{d}{dx}(\cos x) = -\sin x$

(viii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(ix) $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$

(x) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$

(xi) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(xii) $\frac{d}{dx}(\text{constant}) = 0$

(xiii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, $\quad -1 < x < 1$

(xiv) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$, $\quad -1 < x < 1$

(xv) $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$, $\quad x \in R$

(xvi) $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$, $\quad x \in R$

(xvii) $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$, $\quad |x| > 1$

(xviii) $\frac{d}{dx}\left(\text{cosec}^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$, $\quad |x| > 1$

(xix) **Results :**

If the inverse functions f & g are defined by $y = f(x)$ & $x = g(y)$. Then $g(f(x)) = x$.

$\Rightarrow$ $g'(f(x)) \,.\, f'(x) = 1$.

This result can also be written as, if $\frac{dy}{dx}$ exists & $\frac{dy}{dx} \neq 0$, then

$$\frac{dx}{dy} = 1/\left(\frac{dy}{dx}\right) \text{ or } \frac{dy}{dx}.\frac{dx}{dy} = 1 \text{ or } \frac{dy}{dx} = 1/\left(\frac{dx}{dy}\right)\left[\frac{dx}{dy} \neq 0\right]$$

3. THEOREMS ON DERIVATIVES

If u and v are derivable functions of x, then,

(i) Term by term differentiation : $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

(ii) Multiplication by a constant $\frac{d}{dx}(K\,u) = K\frac{du}{dx}$, where K is any constant

(iii) **"Product Rule"** $\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$ known as

In general,

(a) If $u_1, u_2, u_3, u_4, ..., u_n$ are the functions of x, then

$$\frac{d}{dx}(u_1 \,.\, u_2 \,.\, u_3 \,.\, u_4 \,.....\, u_n)$$

$$= \left(\frac{du_1}{dx}\right)(u_2\, u_3\, u_4 \ldots u_n) + \left(\frac{du_2}{dx}\right)(u_1\, u_3\, u_4 \ldots u_n)$$

$$+ \left(\frac{du_3}{dx}\right)(u_1\, u_2\, u_4 \ldots u_n) + \left(\frac{du_4}{dx}\right)(u_1\, u_2\, u_3\, u_5 \ldots u_n)$$

$$+ \ldots + \left(\frac{du_n}{dx}\right)(u_1\, u_2\, u_3 \ldots u_{n-1})$$

(iv) **"Quotient Rule"** $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$ where $v \neq 0$ known as

(b) **Chain Rule :** If $y = f(u)$, $u = g(w)$, $w = h(x)$

then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dw}.\frac{dw}{dx}$

or $\frac{dy}{dx} = f'(u) \,.\, g'(\quad) \,.\, h'(x)$

Note...

In general if $y = f(u)$ then $\frac{dy}{dx} = f'(u).\frac{du}{dx}$.

4. METHODS OF DIFFERENTIATION

4.1 Derivative by using Trigonometrical Substitution

Using trigonometrical transformations before differentiation shorten the work considerably. Some important results are given below :

(i) $\sin 2x = 2 \sin x \cos x = \frac{2\tan x}{1+\tan^2 x}$

(ii) $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \frac{1-\tan^2 x}{1+\tan^2 x}$

(iii) $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$, $\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$

(iv) $\sin 3x = 3\sin x - 4\sin^3 x$

(v) $\cos 3x = 4\cos^3 x - 3\cos x$

(vi) $\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

(vii) $\tan\left(\frac{\pi}{4} + x\right) = \frac{1+\tan x}{1-\tan x}$

(viii) $\tan\left(\frac{\pi}{4}-x\right)=\frac{1-\tan x}{1+\tan x}$

(ix) $\sqrt{(1\pm\sin x)}=\left|\cos\frac{x}{2}\pm\sin\frac{x}{2}\right|$

(x) $\tan^{-1}x\pm\tan^{-1}y=\tan^{-1}\left(\frac{x\pm y}{1\mp xy}\right)$

(xi) $\sin^{-1}x\pm\sin^{-1}y=\sin^{-1}\left\{x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\right\}$

(xii) $\cos^{-1}x\pm\cos^{-1}y=\cos^{-1}\left\{xy\mp\sqrt{1-x^2}\,\sqrt{1-y^2}\right\}$

(xiii) $\sin^{-1}x+\cos^{-1}x=\tan^{-1}x+\cot^{-1}x=\sec^{-1}x+\mathrm{cosec}^{-1}x=\pi/2$

(xiv) $\sin^{-1}x=\mathrm{cosec}^{-1}(1/x)$; $\cos^{-1}x=\sec^{-1}(1/x)$; $\tan^{-1}x=\cot^{-1}(1/x)$

Note...

Some standard substitutions :

Expressions	Substitutions
$\sqrt{(a^2-x^2)}$	$x=a\sin\theta$ or $a\cos\theta$
$\sqrt{(a^2+x^2)}$	$x=a\tan\theta$ or $a\cot\theta$
$\sqrt{(x^2-a^2)}$	$x=a\sec\theta$ or $a\,\mathrm{cosec}\,\theta$
$\sqrt{\left(\frac{a+x}{a-x}\right)}$ or $\sqrt{\left(\frac{a-x}{a+x}\right)}$	$x=a\cos\theta$ or $a\cos 2\theta$
$\sqrt{(a-x)(x-b)}$ or	$x=a\cos^2\theta+b\sin^2\theta$
$\sqrt{\left(\frac{a-x}{x-b}\right)}$ or $\sqrt{\left(\frac{x-}{a-x}\right)}$	
$\sqrt{(x-a)(x-b)}$ or	$x=a\sec^2\theta-b\tan^2\theta$
$\sqrt{\left(\frac{x-a}{x-b}\right)}$ or $\sqrt{\left(\frac{x-}{x-a}\right)}$	
$\sqrt{(2ax-x^2)}$	$x=a(1-\cos\theta)$

4.2 Logarithmic Differentiation

To find the derivative of :

If $y=\{f_1(x)\}^{f_2(x)}$ or $y=f_1(x).f_2(x).f_3(x)...$

or $y=\frac{f_1(x).f_2(x).f_3(x)...}{g_1(x).g_2(x).g_3(x)...}$

then it is convenient to take the logarithm of the function first and then differentiate. This is called derivative of the logarithmic function.

Important Notes (Alternate methods)

1. If $y=\{f(x)\}^{g(x)}=e^{g(x)\ln f(x)}$ ((variable)$^{\text{variable}}$) $\{\because x=e^{\ln x}\}$

$$\therefore \frac{dy}{dx}=e^{g(x)\ln f(x)}.\left\{g(x).\frac{d}{dx}\ln f(x)+\ln f(x).\frac{d}{dx}g(x)\right\}$$

$$=\{f(x)\}^{g(x)}.\left\{g(x).\frac{f'(x)}{f(x)}+\ln f(x).\,g'(x)\right\}$$

2. If $y=\{f(x)\}^{g(x)}$

$\therefore \frac{dy}{dx}=$ Derivative of y treating $f(x)$ as constant + Derivative of y treating $g(x)$ as constant

$$=\{f(x)\}^{g(x)}.\ln f(x).\frac{d}{dx}g(x)+g(x)\{f(x)\}^{g(x)-1}.\frac{d}{dx}f(x)$$

$$=\{f(x)\}^{g(x)}.\ln f(x).g'(x)+g(x).\{f(x)\}^{g(x)-1}.f'(x)$$

4.3 Implict Differentiation : $\phi(x, y)=0$

(i) In order to find dy/dx in the case of implicit function, we differentiate each term w.r.t. x, regarding y as a function of x & then collect terms in dy/dx together on one side to finally find dy/dx.

(ii) In answers of dy/dx in the case of implicit function, both x & y are present.

Alternate Method : If $f(x, y)=0$

then $$\frac{dy}{dx}=-\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}=-\frac{\text{diff. of f w.r.t. x treating y as constant}}{\text{diff. of f w.r.t. y treating x as constant}}$$

4.4 Parametric Differentiation

If $y = f(t)$ & $x = g(t)$ where t is a Parameter, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad ...(1)$$

Note...

1. $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

2. $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx} \quad \left(\because \frac{dy}{dx} \text{ in terms of t}\right)$

$$= \frac{d}{dt}\left(\frac{f'(t)}{g'(t)}\right) \cdot \frac{1}{f'(t)} \quad \{\text{From}(1)\}$$

$$= \frac{f''(t)g'(t) - g''(t)f'(t)}{\{f'(t)\}}$$

4.5 Derivative of a Function w.r.t. another Function

Let $y = f(x)$; $z = g(x)$ then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$

4.6 Derivative of Infinite Series

If taking out one or more than one terms from an infinite series, it remains unchanged. Such that

(A) If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + ...\infty}}}$

then $y = \sqrt{f(x) + y} \Rightarrow (y^2 - y) = f(x)$

Differentiating both sides w.r.t. x, we get $(2y - 1)\frac{dy}{dx} = f'(x)$

(B) If $y = \{f(x)\}^{\{f(x)\}^{\{f(x)\}^{.......\infty}}}$ then $y = \{f(x)\}^y \Rightarrow y = e^{y \ln f(x)}$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{y\{f(x)\}^{y-1} \cdot f'(x)}{1 - \{f(x)\}^y \cdot \ell n\, f(x)} = \frac{y^2 f'(x)}{f(x)\{1 - y\, \ell n\, f(x)\}}$$

5. DERIVATIVE OF ORDER TWO & THREE

Let a function $y = f(x)$ be defined on an open interval (a, b). It's derivative, if it exists on (a, b), is a certain function $f'(x)$ [or (dy/dx) or y′] & is called the first derivative of y w.r.t. x. If it happens that the first derivative has a derivative on (a, b) then this derivative is called the second derivative of y w.r.t. x & is denoted by $f''(x)$ or (d^2y/dx^2) or y″.

Similarly, the 3rd order derivative of y w.r.t. x, if it exists, is defined by $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ it is also denoted by $f'''(x)$ or y‴.

Some Standard Results :

(i) $\frac{d^n}{dx^n}(ax+b)^m = \frac{m!}{(m-n)!} \cdot a^n \cdot (ax+b)^{m-n}$, $m \geq n$.

(ii) $\frac{d^n}{dx^n} x^n = n!$

(iii) $\frac{d^n}{dx^n}\left(e^{mx}\right) = m^n \cdot e^{mx}$, $m \in R$

(iv) $\frac{d^n}{dx^n}(\sin(ax+b)) = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$, $n \in N$

(v) $\frac{d^n}{dx^n}(\cos(ax+b)) = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$, $n \in N$

(vi) $\frac{d^n}{dx^n}\{e^{ax}\sin(bx+c)\} = r^n \cdot e^{ax} \cdot \sin(bx + c + n\phi)$, $n \in N$

where $r = \sqrt{(a^2 + b^2)}$, $\phi = \tan^{-1}(b/a)$.

(vii) $\frac{d^n}{dx^n}\{e^{ax} \cdot \cos(bx+c)\} = r^n \cdot e^{ax} \cdot \cos(bx + c + n\phi)$, $n \in N$

where $r = \sqrt{(a^2 + b^2)}$, $\phi = \tan^{-1}(b/a)$.

6. DIFFERENTIATION OF DETERMINANTS

If $F(X)=\begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$,

where *f, g, h, ℓ, m, n, u, v, w* are differentiable function of x then

$$F'(x)=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x) \\ \ell'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$$

$$+\begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

7. L' HOSPITAL'S RULE

If $f(x)$ & $g(x)$ are functions of x such that :

(i) $\lim_{x\to a} f(x)=0=\lim_{x\to a} g(x)$ or $\lim_{x\to a} f(x)=\infty=\lim_{x\to a} g(x)$ and

(ii) Both $f(x)$ & $g(x)$ are continuous at x = a and

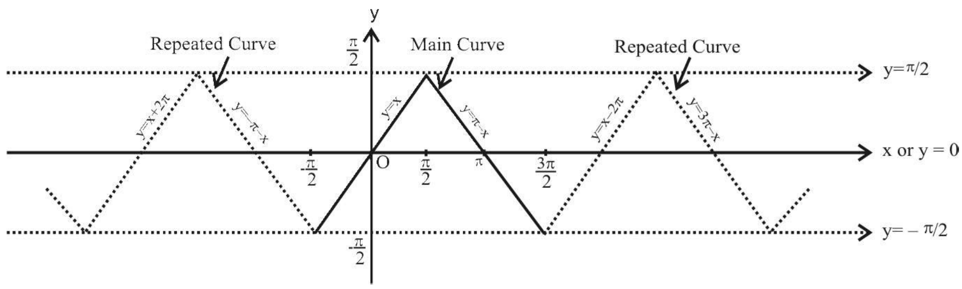
(iii) Both $f(x)$ & $g(x)$ are differentiable at x = a and

(iv) Both $f'(x)$ & $g'(x)$ are continuous at x = a, Then

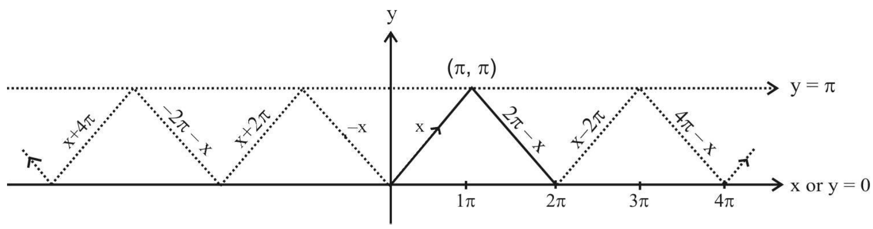
$$\underset{x\to a}{\text{Limit}}\frac{f(x)}{g(x)}=\underset{x\to a}{\text{Limit}}\frac{f'(x)}{g'(x)}=\underset{x\to a}{\text{Limit}}\frac{f''(x)}{g''(x)}$$ & so on till indeterminant form vanishes..

8. ANALYSIS & GRAPHS OF SOME USEFUL FUNCTION

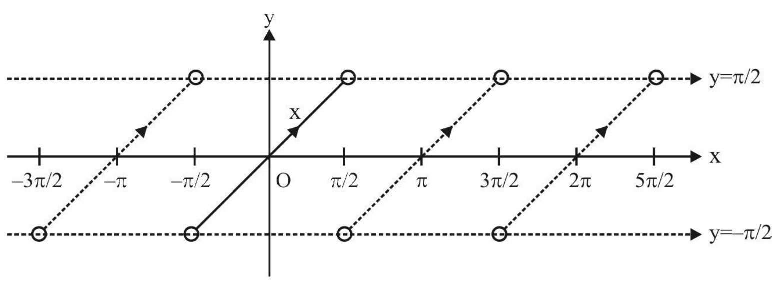
(i) $y=\sin^{-1}(\sin x)$ $\quad x\in R\,;\; y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$



(ii) $y=\cos^{-1}(\cos x)$ $\quad x\in R\,;\; y\in[0,\pi]$



(iii) $y=\tan^{-1}(\tan x)$ $\quad x\in R-\left\{x: x=(2n+1)\frac{\pi}{2}, n\in Z\right\};\; y\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$



1. $\lim_{x\to 0} \frac{\sin x}{x(1+\cos x)}$ is equal to

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) -1

2. $\lim_{x\to \frac{\pi}{2}} \frac{1-\sin x}{\cos x}$ is equal to :

(a) 0 (b) -1 (c) 1 (d) Does not exist

3. $\lim_{x\to 0} \frac{|x|}{x}$ is equal to

(a) 1 (b) -1 (c) 0 (d) Does not exist

4. $\lim_{x\to 1}[x-1]$ where $[\cdot]$ is the greatest integer function is equal to:

(a) 1 (b) 2 (c) 0 (d) Does not exist

5. $\lim_{x\to 0} x\ sin\frac{1}{x}$ is equal to:

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) Does not exist

6. $\lim_{n\to\infty} \frac{1+2.+3+\cdots+n}{n^2}, n \in N$ is equal to :

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

7. $\lim_{x\to 0} \frac{\sin x}{\sqrt{x+1}-\sqrt{1-x}}$ is:

(a) 2 (b) 0 (c) 1 (d) -1

8. $\lim_{x\to 1} \frac{(\sqrt{x}-1)(2x-3)}{2x^2+x-3}$ is

(a) $\frac{1}{10}$ (b) $-\frac{1}{10}$ (c) 1 (d) Does not exist

9. Let $f(x) = \begin{cases} x^2-1, 0<x<2 \\ 2x+3, 2\le x<3 \end{cases}$, the quadratic equation Whose roots are $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$ is:

(a) $x^2-6x+9=0$ (b) $x^2-7x+8=0$ (c) $x^2-14x+49=0$ (d) $x^2-10x+21=0$

10. Differential coefficient of $\sec(tan^{-1}x)$ w.r.t x is:

(a) $\frac{x}{\sqrt{1+x^2}}$ (b) $\frac{x}{1+x^2}$ (c) $x\sqrt{1+x^2}$ (d) $\frac{1}{\sqrt{1+x^2}}$

11. if $u = sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v = tan^{-1}\left(\frac{2x}{1-x^2}\right)$ then $\frac{du}{dv}$ is:

(a) $\frac{1}{2}$ (b) x (c) $\frac{1-x^2}{1+x^2}$ (d) 1

12. if $y = \sqrt{\sin x + y}, then\ \frac{dy}{dx}$ is equal to:

(a) $\frac{\cos x}{2y-1}$ (b) $\frac{\cos x}{1-2y}$ (c) $\frac{\sin x}{1-2y}$ (d) $\frac{\sin x}{2y-1}$

13.The derivative of $cos^{-1}(2x^2-1)$ w.r.t $cos^{-1}x$ is

(a) 2 (b) $\frac{-1}{2\sqrt{1-x^2}}$ (c) $\frac{2}{x}$ (d) 1-x^2

14. if x= t^2 , y= t^3 then $\frac{d^2y}{dx^2}$ is

(a) $\frac{3}{2}$ (b) $\frac{3}{4t}$ (c) $\frac{3}{2t}$ (d) 3

15. if f(x)=e^x g(x), g(0) =2, g'(0)=1 then f '(0) is:

(a) 1 (b) 3 (c) 2 (d) 0

16.if f(x+y)=f(x)f(y) for all x and y and f(5)=2, f(0)=3 then f '(5) is:

(a) 5 (b) 6 (c) 0 (d) None of these

17. if $x^y = e^{x-y}$ *then* $\frac{dy}{dx}$ is equal to :

(a) $\frac{y}{(1+\log x)^2}$ (b) $\frac{x}{(1+\log x)^2}$ (c) $\frac{\log x}{(1+\log x)^2}$ (d) None of these

18. if $y = sec^{-1}\left(\frac{x+1}{x-1}\right) + sin^{-1}\left(\frac{x-1}{x+1}\right)$ *then* $\frac{dy}{dx}$ *is* :

(a) 1 (b) $\frac{x-1}{x+1}$ (c) $\frac{x+1}{x-1}$ (d) 0

19. The value of . $\lim_{x\to 0}(\cos x + \text{a}\sin bx)^{\frac{1}{x}}$ is

(a) 1 (b) ab (c) e^{ab} (d) $e^{\frac{a}{b}}$

20. $\lim_{h\to 0}\frac{cos^2(x+h)-cos^2x}{h}$ is equal to :

(a) cos^2x (b) –sin $2x$ (c) sin x cos x (d) 2sin x

21. The derivative of $sin^{-1}x$ w.r.t. $cos^{-1}\sqrt{1-x^2}$ is

(a) $\frac{1}{\sqrt{1-x^2}}$ (b) 1 (c) $cos^{-1}x$ (d) $tan^{-1}\frac{x}{\sqrt{1-x^2}}$

22. *if* $2^x + 2^y = 2^{x+y}$ *then* $\frac{dy}{dx}$ *is equal to*:

(a) $\frac{2^x+2^y}{2^x-2^y}$ (b) $\frac{2^x+2^y}{1+2^{x+y}}$ (c) $2^{x-y}\left(\frac{2^y-1}{1-2^x}\right)$ (d) $\frac{2^{x+y}-2^x}{2^y}$

23. *if* $y = \sqrt{x+\sqrt{x+\sqrt{x+\cdots to\ \infty}}}$ *then* $\frac{dy}{dx}$ is equal

(a) 1 (b) $\frac{1}{xy}$ (c) $\frac{1}{2y-x}$ (d) $\frac{1}{2y-1}$

24. *if* $y = \sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\cdots to\ \infty}}}$ then the value of $\frac{dy}{dx}$ *is equal to*

(a) $\sqrt{\frac{\sin x}{y+1}}$ (b) $\frac{\sin x}{y+1}$ (c) $\frac{\cos x}{2y+1}$ (d) $\frac{\cos x}{2y-1}$

25. if $f(x) = log_{e^2}(\log x)$ $then$ $f'(e) is$:

(a) 0 (b) $\frac{1}{2e}$ (c) $\frac{e}{2}$ (d) $\frac{2}{e}$

26. $f(x) = cot^{-1}(\cos 2x)^{\frac{1}{2}}$ $then$ $f'(\frac{\pi}{6})$ is:

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{2}{\sqrt{3}}$ (c) $\sqrt{\frac{2}{3}}$ (d) $-\frac{2}{\sqrt{3}}$

27. $\frac{d}{dx}\left[cosec^{-1}\left(\frac{1+x^2}{2x}\right)\right]$ is

(a) $-\frac{2}{1+x^2}, x \neq 0$ (b) $\frac{2}{1+x^2}, x \neq 0$ (c) $\frac{2(1-x^2)}{(1+x^2)|1-x^2|}, x \neq \pm 1,0$ (d) None of these

28. if $y = \log(\sqrt{x} + \sqrt{x-a})$ $then$ $\frac{dy}{dx}$ is

(a) $\frac{1}{\sqrt{x}+\sqrt{x-a}}$ (b) $\frac{1}{2\sqrt{x}\sqrt{x-a}}$ (c) $\frac{1}{\sqrt{x}\sqrt{x-a}}$ (d) None of these

ANSWERS

1	**b**	2	**a**	3	**d**	4	**d**	5	**a**	6	**c**	7	**c**	8	**b**	9	**d**
10	**a**	11	**d**	12	**a**	13	**a**	14	**b**	15	**b**	16	**b**	17	**c**	18	**d**
19	**c**	20	**b**	21	**b**	22	**c**	23	**d**	24	**d**	25	**b**	26	**c**	27	**b**
28	**b**																